PROBLEM SET II, PROBLEMS III, IV

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Problem 1. Let E be a subset of $\mathbb{R}^n$. Show that E is measurable iff $\mu^\star(B \cap E) + \mu^\star(B \cap E^c) = \mu^\star(B)$ for every open box $B \subseteq \mathbb{R}^n$.

Proof. The forward direction proceeds definitionally.

Conversely, suppose that

$$\mu^\star(B \cap E) + \mu^\star(B \cap E^c) = \mu^\star(B)$$

for every open box $B \subseteq \mathbb{R}^n$. We claim that E is measurable; that is, if S is an arbitrary subset of $\mathbb{R}^n$, then

$$\mu^\star(S \cap E) + \mu^\star(S \cap E^c) = \mu^\star(S).$$

By sub-additivity

$$\mu^\star(S \cap E) + \mu^\star(S \cap E^c) \geq \mu^\star(S).$$

We now prove the reverse inequality. Assume that $\mu^\star(S) < \infty$. Select $\epsilon > 0$. From our definition of outer measure, we can find open boxes $\{B_1, B_2, B_3, ...\}$ such that

$$S \leq \bigcup_{i=1}^{\infty} B_i$$

and

$$\sum_{i=1}^{\infty} \operatorname{vol}(B_i) \leq \mu^\star(S) + \epsilon.$$

Then

$$\begin{aligned}\mu^\star(S \cap E) &\leq \mu^\star\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right)\\ &\leq \mu^\star\left(\bigcup_{i=1}^{\infty} (E \cap B_i)\right)\\ &\leq \sum_{i=1}^{\infty} \mu^\star(E \cap B_i),\end{aligned}$$

where we justify the first inequality by monotonicity and the last by sub-additivity. Similarly,

$$\begin{aligned}\mu^\star(S \cap E^c) &\leq \left(E^c \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right)\\ &\leq \mu^\star\left(\bigcup_{i=1}^{\infty} (E^c \cap B_i)\right)\end{aligned}$$

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$$\leq \sum_{i=1}^{\infty} \mu^{\star}\left(E^{c} \cap B_{i}\right),$$

where we again justify the first inequality by monotonicity and the last by subadditivity. Combining these yields

$$\mu^{\star}(S \cap E)+\left(S \cap E^{c}\right) \leq \sum_{i=1}^{\infty} \mu^{\star}\left(E \cap B_{i}\right)+\sum_{i=1}^{\infty} \mu^{\star}\left(E^{c} \cap B_{i}\right).$$

The RHS may be added termwise by the absolute convergence of the series. We observe that, since $\mu^{\star}\left(E \cap B_{i}\right)+\mu^{\star}\left(E^{c} \cap B_{i}\right)=\mu^{\star}\left(B_{i}\right)$, we have

$$\sum_{i=1}^{\infty} \mu^{\star}\left(E \cap B_{i}\right)+\sum_{i=1}^{\infty} \mu^{\star}\left(E^{c} \cap B_{i}\right)=\sum_{i=1}^{\infty} \mu^{\star}\left(B_{i}\right).$$

Thus,

$$\mu^{\star}(S \cap E)+\mu^{\star}\left(S \cap E^{c}\right) \leq \sum_{i=1}^{\infty} \mu^{\star}\left(B_{i}\right) \leq \mu^{\star}(S)+\epsilon.$$

Our selection of $\epsilon>0$ was arbitrary, so we have the desired inequality:

$$\mu^{\star}(S \cap E)+\mu^{\star}\left(S \cap E^{c}\right) \leq \mu^{\star}(S).$$

Thus, our proof of the converse is complete, and we are done. □

Problem 2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Show that

$$S=\{x \in \mathbb{R}: f \text{ differentiable at } x\}$$

is a Borel set.

Proof. A function f is differentiable at a point x precisely when $\lim_{h \rightarrow 0} F(x, h)$ exists, where $F: \mathbb{R} \times(\mathbb{R} /\{0\}) \rightarrow \mathbb{R}$. Putting this in terms of the usual δ, ϵ formalization:

$$\exists l: \forall \epsilon>0: \exists \delta>0: \forall|h|<\delta,$$
$$\left|\frac{f(x+h)-f(x)}{h}-l\right|<\epsilon.$$

However, this formalization is not particularly helpful in our case, as the variables are assumed to take real values. We wish to show that the set of points where f is differentiable is a Borel set; that is, we wish to deal with countable sets closed under countable unions and intersections. Thus, we modify our formalization of the limit so that we might restrict ϵ, δ, h, l to $\mathbb{Q}$ (ensuring countability). I claim that the classical formalization is equivalent to

$$\forall \epsilon>0: \exists \delta>0: \exists L: \forall|h|<\delta,$$
$$\left|\frac{f(x+h)-f(x)}{h}-L\right|<\epsilon.$$

The first statement trivially implies the second. To prove the reverse direction, define

$$F(x, h)=\frac{f(x+h)-f(x)}{h},$$

and select δ, L such that

$$|F(x, h)-L|<\frac{\epsilon}{2}, \forall|h|<\delta.$$

Then, for δ sufficiently small

$$|F(x,h) - F(x,c)| < \epsilon$$

with $|h|, |c| < \delta$, which we justify by the Triangle Inequality. This implies that the limit of our difference quotient is Cauchy. By the completeness of $\mathbb{R}$, the limit of $F(x,h)$ exists.

Using our new formalization of the limit, ϵ, δ, L may be restricted to the rationals. By the continuity of the difference quotient, h may also be restricted to $\mathbb{Q}$. The rest of the solution amounts to parsing the quantifiers in our definition.

By the continuity of F, the set of x for which $|F(x,h) - L| < \epsilon$ is an open set, and then Borel. Given ϵ, δ, L, the set of x such that

$$|F(x,h) - L| < \epsilon, \ \forall |h| < \delta$$

is equivalent to

$$\bigcap_h \{x : |F(x,h) - L| < \epsilon\}.$$

Thus, we have our above expression is a countable intersection of Borel sets and hence Borel. Working outwards with our quantifiers, we now consider the set of x, given some ϵ, δ, such that

$$\exists L : \forall h : |h| < \delta, |F(x,h) - L| < \epsilon.$$

This is the countable union, over all possible values of L, of Borel sets. Hence, it is Borel.

Continuing in this fashion, we extend our argument to ϵ, δ, and see that the set of points for which f is differentiable is a Borel set, as desired.

This is all well and good, but parsing through quantifiers can be somewhat odious. We might also try a slightly different approach.

Since we know that $\mathbb{R}$ is complete, let us formulate the problem in terms of the Cauchy definition of a limit. Write the set S of points x where f is differentiable as

$$S = \left\{x : \forall n \in \mathbb{N}, \exists N \in \mathbb{N} : N < q_1, q_2 \in \mathbb{N} : \left|F_{q_1}(x) - F_{q_2}(x)\right| < \frac{1}{n}\right\}$$

where we define

$$F_q(x) = \frac{f\left(x + \frac{1}{q}\right) - f(x)}{\frac{1}{q}}.$$

Now let us consider the set

$$S_{n,q_1,q_2} = \left\{x \in \mathbb{R} : \left|F_{q_1}(x) - F_{q_2}(x)\right| < \frac{1}{n}\right\}.$$

We claim that this set is open. Indeed, by the continuity of f, and thus the continuity of F_q, the set S_{n,q_1,q_2} is the pre-image of an open set under a continuous function. Thus, we may write

$$S = \bigcap_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{\substack{q_1, q_2 \in \mathbb{N} \\ q_1, q_2 > N}} S_{n,q_1,q_2}.$$

From this, we may conclude that S is Borel. □